Manuel Stein

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WARTIME REPORT

CRITICAL COMBINATIONS OF LONGITUDINAL AND TRANSVERSE

DIRECT STRESS FOR AN INFINITELY LONG FLAT PLATE WITH

EDGES ELASTICALLY RESTRAINED AGAINST ROTATION

By S. B. Batdorf, Manuel Stein, and Charles Libove

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

NACA

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

ADVANCE RESTRICTED REPORT

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SUMMARY

A theoretical investigation was made of the buckling
of an infinitely long flat plate with edges elastically
restrained against rotation under combinations of longi-
tudinal and transverse direct stress. Interaction curves
are presented that give the critical combinations of
stress for several different degrees of elastic edge
restraint, including simple support and complete fixity.
It was found that an appreciable fraction of the critical
longitudinal stress may be applied to the plate without
any reduction in the transverse compressive stress
required for buckling.

INTRODUCTION

Because the skin of an airplane in flight is sub-
jected to combinations of stress, attention has recently
been given to the problem of plate buckling when more
than one stress is acting. The present paper is the
third of a series of papers analyzing the elastic buck-
ling, under the action of two stresses, of an infinitely
long flat plate with edges equally restrained against
rotation and fully supported. The two previous papers
are reference 1, which deals with the interaction of
shear and longitudinal direct stress, and reference 2,
which deals with the interaction of shear and transverse
direct stress. The present paper describes the inter-
action of longitudinal and transverse direct stress.
These three loading combinations are illustrated in
figure 1.

   Interaction curves that give the critical direct-
stress combinations for several different degrees of
elastic edge restraint, including simple support and
complete fixity, are presented for the case in which the
magnitude of the restraint is independent of the buckle
wave length. These curves are based on an exact solu-
tion of the differential equation of equilibrium, the
details of which are given in the appendix.

SYMBOLS

E   elastic modulus of plate material

μ Poisson's ratio for plate material

t   thickness of plate

b   width of plate

a   length of plate $(a > b)$

D   flexural stiffness of plate per unit length

$$\left[\frac{Et^3}{12\left(1 - \mu^2\right)} \right]$$

x   longitudinal coordinate

y   transverse coordinate

w   normal displacement of a point on buckled plate
             from its undeflected position

λ half wave length of buckle

S_o rotational stiffness of restraining medium along
             edges of plate, moment per quarter radian per
             unit length

ϵ dimensionless elastic edge-restraint constant
             $\left(\dfrac{4S_o b}{D} \right)$

σ_x applied uniform longitudinal compressive stress

σ_y applied uniform transverse compressive stress

$N_x = \sigma_x t$

$N_y = \sigma_y t$

k_x, k_y dimensionless stress coefficients $\left(k_x = \dfrac{\sigma_x b^2 t}{\pi^2 D}; \right.$

$$\left. k_y = \frac{\sigma_y b^2 t}{\pi^2 D} \right)$$

R_x longitudinal direct-stress ratio; ratio of longitudinal direct stress present to critical stress in pure longitudinal compression

R_y transverse direct-stress ratio; ratio of transverse direct stress present to critical stress in pure transverse compression

RESULTS AND DISCUSSION

The results of this investigation are given in the form of nondimensional interaction curves in figure 2. Each point on these curves represents a critical combination of the stress coefficients k_x and k_y for a given elastic edge-restraint constant ϵ at which an infinitely long flat plate will buckle. The interaction curves for plates with simply supported and clamped edges are given in figure 3 in terms of stress ratios rather than stress coefficients. The calculated data used to plot the interaction curves are given in table 1.

Applicability of the interaction curves.- Critical combinations of longitudinal and transverse direct stress for an infinitely long flat plate with edges either simply supported or clamped can be obtained from the interaction curves of figure 2. Critical combinations of direct stress for a plate with intermediate elastic restraint against edge rotation can also be obtained from figure 2 for those cases in which the stiffness of the restraining medium is independent of buckle wave length (ϵ = a constant). Such edge restraint is provided

only by a medium in which rotation at one point does not
influence rotation at another point. Edge conditions of
this type are not ordinarily encountered but might occur
when the restraint is furnished by a row of discrete
elements, such as coil springs or flexible clamps.
Because of the great variety of possible relationships
between edge restraint and wave length, only the curves
for edge restraint independent of wave length are shown.
If critical stress combinations for a plate with con-
tinuous edge restraint (ϵ dependent on wave length) are
desired, they can be computed - though somewhat labo-
riously - by the method outlined in the appendix, pro-
vided the relationship between edge restraint and wave
length is known. This relationship is derived in refer-
ence 3 for the special case of a sturdy stiffener, that
is, a stiffener which twists without cross-sectional
distortion.

The buckling stress for a finite plate can never be
lower than that for an infinite plate having the same
width and thickness because the finite plate is strength-
ened by support along two additional edges. The use of
figure 2 to estimate the critical direct stresses for a
finite plate with edge restraint independent of wave
length, therefore, is in all cases conservative.

Vertical portions of interaction curves.- The
vertical portions of the interaction curves (fig. 2)
indicate that a considerable amount of longitudinal com-
pression may be applied to the plate without any reduc-
tion in the transverse compression required for buckling.
This result parallels the result of reference 2, in
which it was found that a considerable amount of shear
stress could be applied to an infinitely long plate
without any reduction in the transverse compression
necessary to cause buckling. On the other hand, in ref-
erence 1 it was shown that the presence of shear always
reduces the longitudinal compressive stress required to
produce buckles. This disparity in behavior is probably
attributable to the character of the buckle forms for
the three types of stress. (See fig. 4.) The buckle
form for shear alone (fig. 4(a)) can be transformed con-
tinuously into that for longitudinal compression alone
(fig. 4(b)) by a gradual addition of compression and
subtraction of shear. Neither of these buckle forms,
however, can be continuously transformed into the buckle
form for transverse compression alone (fig. 4(c)).

The vertical portions of the interaction curves extend indefinitely into the tension region of k_x. (For convenience, in fig. 2 the curves are stopped at a small negative value of k_x.) This property of the curves indicates that the presence of longitudinal tension has no effect upon the transverse stress necessary to produce buckling.

SUMMARY OF RESULTS

Interaction curves are presented from which critical combinations of longitudinal and transverse direct stress for an infinitely long flat plate with edges either simply supported or clamped can be obtained. Critical combinations of direct stress for intermediate elastic restraint against edge rotation can also be obtained from the interaction curves for those cases in which the stiffness of the restraining medium is independent of buckle wave length. For cases in which the stiffness of the restraining medium depends upon the buckle wave length and the relationship between the two is known, the critical combinations of direct stress can be determined - though somewhat laboriously - by a method similar to that used in obtaining the interaction curves.

A considerable amount of longitudinal compression may be applied to an infinitely long flat plate before there is any reduction in the transverse compression necessary to produce buckling. The presence of longitudinal tension has no effect upon the transverse stress necessary to produce buckling.

The use of the interaction curves to determine the critical stresses for a finite plate with edge restraint independent of wave length is in all cases conservative.

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APPENDIX

BUCKLING OF INFINITELY LONG PLATES

UNDER TWO DIRECT STRESSES

Differential equation of equilibrium.- The critical combinations of longitudinal and transverse direct stress that will cause buckling in an infinitely long flat plate with edges elastically restrained against rotation can be obtained by solving the differential equation of equilibrium. This equation, adapted from page 324 of reference 4, is

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} = 0 \qquad (A1)$$

where N_x and N_y are positive for compression. (The coordinate system used is given in fig. 5.) Equation (A1) may be rewritten and used in the following form:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + k_x\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x^2} + k_y\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial y^2} = 0 \qquad (A2)$$

where

$$k_x = \frac{N_x b^2}{\pi^2 D}$$

and

$$k_y = \frac{N_y b^2}{\pi^2 D}$$

Solution of differential equation.- If the plate is infinitely long in the x-direction, all displacements are periodic in x and the buckled surface is assumed to have the form

$$w = Y\cos\frac{\pi x}{\lambda} \qquad (A3)$$

where Y is a function of y only and λ is the half wave length of the buckle in the x-direction. Substitution

of the expression for w given in equation (A3) in
the differential equation (A2) yields the following
equation:

$$\frac{d^4Y}{dy^4} - \frac{2\pi^2}{\lambda^2}\frac{d^2Y}{dy^2} + \frac{\pi^4}{\lambda^4}Y - \frac{\pi^4}{b^2\lambda^2}k_xY + \frac{\pi^2}{b^2}k_y\frac{d^2Y}{dy^2} = 0 \quad (A4)$$

Equation (A4) must be satisfied by Y if the assumed
deflection is to satisfy the differential equation (A2).
The expression

$$Y = e^{\frac{imy}{b}} \quad (A5)$$

will be a solution of equation (A4) when m is a root
of the characteristic equation

$$m^4 + \left(2\frac{b^2}{\lambda^2} - k_y\right)\pi^2m^2 + \pi^4\frac{b^2}{\lambda^2}\left(\frac{b^2}{\lambda^2} - k_x\right) = 0 \quad (A6)$$

The roots of this equation are

$$\left.\begin{array}{l} m_1 = \pi\sqrt{\dfrac{k_y}{2} - \dfrac{b^2}{\lambda^2} + \dfrac{1}{2}\sqrt{k_y{}^2 + 4\dfrac{b^2}{\lambda^2}(k_x - k_y)}} \\[2em] m_2 = -\pi\sqrt{\dfrac{k_y}{2} - \dfrac{b^2}{\lambda^2} + \dfrac{1}{2}\sqrt{k_y{}^2 + 4\dfrac{b^2}{\lambda^2}(k_x - k_y)}} \\[2em] m_3 = \pi\sqrt{\dfrac{k_y}{2} - \dfrac{b^2}{\lambda^2} - \dfrac{1}{2}\sqrt{k_y{}^2 + 4\dfrac{b^2}{\lambda^2}(k_x - k_y)}} \\[2em] m_4 = -\pi\sqrt{\dfrac{k_y}{2} - \dfrac{b^2}{\lambda^2} - \dfrac{1}{2}\sqrt{k_y{}^2 + 4\dfrac{b^2}{\lambda^2}(k_x - k_y)}} \end{array}\right\} \quad (A7)$$

The complete solution of equation (A4) is therefore

$$Y = Pe^{\frac{im_1y}{b}} + Qe^{\frac{im_2y}{b}} + Re^{\frac{im_3y}{b}} + Se^{\frac{im_4y}{b}} \quad (A8)$$

where P, Q, R, and S are constants to be determined
from the boundary conditions.

The solution of the differential equation (A2) can now be written

$$w = \left(Pe^{\frac{im_1 y}{b}} + Qe^{\frac{im_2 y}{b}} + Re^{\frac{im_3 y}{b}} + Se^{\frac{im_4 y}{b}}\right)\cos\frac{\pi x}{\lambda} \qquad (A9)$$

Stability criterion.- The boundary conditions that must be satisfied by the solution of the differential equation of equilibrium are

$$
\left.\begin{aligned}
(Y)_{y=\frac{b}{2}} &= 0 \\[2ex]
(Y)_{y=-\frac{b}{2}} &= 0 \\[2ex]
D\left(\frac{d^2 Y}{dy^2}\right)_{y=\frac{b}{2}} &= -4S_0\left(\frac{dY}{dy}\right)_{y=\frac{b}{2}} \\[2ex]
D\left(\frac{d^2 Y}{dy^2}\right)_{y=-\frac{b}{2}} &= 4S_0\left(\frac{dY}{dy}\right)_{y=-\frac{b}{2}}
\end{aligned}\right\} \qquad (A10)
$$

The first two conditions result from the requirement of zero deflection along the edges. The last two conditions express the requirement that the curvature at any point along the edge of the plate be consistent with the transverse bending moment at the point.

If the conditions given in equations (A10) are imposed upon equation (A8), four linear homogeneous equations in P, Q, R, and S result. These equations are

$$Pe^{\frac{im_1}{2}} + Qe^{\frac{im_2}{2}} + Re^{\frac{im_3}{2}} + Se^{\frac{im_4}{2}} = 0$$

$$Pe^{\frac{-im_1}{2}} + Qe^{\frac{-im_2}{2}} + Re^{\frac{-im_3}{2}} + Se^{\frac{-im_4}{2}} = 0$$

$$m_1{}^2 Pe^{\frac{im_1}{2}} + m_2{}^2 Qe^{\frac{im_2}{2}} + m_3{}^2 Re^{\frac{im_3}{2}} + m_4{}^2 Se^{\frac{im_4}{2}}$$
$$-i\epsilon\left(m_1 Pe^{\frac{im_1}{2}} + m_2 Qe^{\frac{im_2}{2}} + m_3 Re^{\frac{im_3}{2}} + m_4 Se^{\frac{im_4}{2}}\right) = 0$$

$$m_1{}^2 Pe^{\frac{-im_1}{2}} + m_2{}^2 Qe^{\frac{-im_2}{2}} + m_3{}^2 Re^{\frac{-im_3}{2}} + m_4{}^2 Se^{\frac{-im_4}{2}}$$
$$+ i\epsilon\left(m_1 Pe^{\frac{-im_1}{2}} + m_2 Qe^{\frac{-im_2}{2}} + m_3 Re^{\frac{-im_3}{2}} + m_4 Se^{\frac{-im_4}{2}}\right) = 0$$

(A11)

where

$$\epsilon = \frac{4S_o b}{D}$$

In order for P, Q, R, and S to have values other than zero, that is, in order for the plate to buckle, the determinant formed by the coefficients of P, Q, R, and S in equations (A11) must equal zero. The expansion of this determinant is given on page 13 of reference 5 for the case in which the roots of the characteristic equation are of the form

$$\frac{m_1}{2} = \gamma + \beta$$

$$\frac{m_2}{2} = \gamma - \beta$$

$$\frac{m_3}{2} = -\gamma + i\alpha$$

$$\frac{m_4}{2} = -\gamma - i\alpha$$

(A12)

In the present problem, the roots (equations (A7)) of
the characteristic equation have the form of equa-
tions (A12), where

$$\gamma = 0$$

$$\beta = \frac{\pi}{2}\sqrt{\left(\frac{k_y}{2} - \frac{b^2}{\lambda^2}\right) + \frac{1}{2}\sqrt{k_y^2 + 4\frac{b^2}{\lambda^2}\left(k_x - k_y\right)}}$$

$$\alpha = \frac{\pi}{2}\sqrt{-\left(\frac{k_y}{2} - \frac{b^2}{\lambda^2}\right) + \frac{1}{2}\sqrt{k_y^2 + 4\frac{b^2}{\lambda^2}\left(k_x - k_y\right)}}$$

$$\left.\begin{array}{c}\end{array}\right\} \quad \text{(A13)}$$

Substitution of $\gamma = 0$ in the stability criterion
given as equation (A19) of reference 5 yields a
stability criterion that is applicable to the present
problem. This stability criterion is

$$\left[\left(\alpha^2 + \beta^2\right)^2 + \left(\alpha^2 - \beta^2\right)\frac{\epsilon^2}{4}\right]\sinh 2\alpha \sin 2\beta$$

$$- 2\alpha\beta\frac{\epsilon^2}{4}\left(\cosh 2\alpha \cos 2\beta - 1\right)$$

$$+ \epsilon\left[\alpha\left(\alpha^2 + \beta^2\right)\cosh 2\alpha \sin 2\beta - \beta\left(\alpha^2 + \beta^2\right)\sinh 2\alpha \cos 2\beta\right] = 0$$

$$\text{(A14)}$$

where α and β are defined in equations (A13). Any
combination of values k_x, k_y, b/λ, and ϵ that satisfies
equation (A14) will cause the plate to be on the point
of buckling.

   Interaction curves for restraint independent of
wave length.- The procedure for plotting interaction
curves is as follows: For a given value of ϵ, a value
of k_y is chosen. Substitution of these values of k_y
and ϵ in equation (A14) yields an equation in terms
of k_x and b/λ. A plot of k_x against b/λ is then
made. Every point on this curve represents a combination
of k_x and b/λ that will maintain neutral equilibrium
for the given value of ϵ and the chosen value of k_y.

Since the plate will buckle at the lowest value of k_x
that will maintain neutral equilibrium, only the minimum
value of k_x is taken from the plot of k_x against b/λ.
This process is repeated for other assumed values of k_y,
and each time a minimum value of k_x is determined.
Finally, the interaction curve of k_y against the minimum
value of k_x can be plotted for the given value of ϵ.

For the special case of a plate with simply supported
edges ($\epsilon = 0$), equation (A14) is simplified to such an
extent that the minimization of k_x with respect to b/λ
can readily be done analytically. The equation of the inter-
action curve for $\epsilon = 0$ can then be given explicitly as

$$k_x = 2\left(1 + \sqrt{1 - k_y}\right) \qquad (A15)$$

The plotting procedure just discussed and the
analytical solution for the case of simply supported
edges (equation (A15)) give only the curved portions of
the interaction curves. The conclusion that the vertical
portions also represent critical stress combinations and
are therefore properly a part of the interaction curves
depends upon an argument analogous to that at the end of
appendix B of reference 2. This argument is based on
the fact that the end point of the curved portion of each
curve can be shown to represent a combination of stresses
for which the buckle wave length is infinite. When the
wave length is infinite, the longitudinal stress can do
no work during buckling. Accordingly, the transverse
stress required to produce buckling is the same as it
would be in the absence of longitudinal compression.
Inasmuch as a reduction in longitudinal compression tends
to increase rather than to diminish the wave length, the
same argument applies to all points on a vertical line
below the end point of the curved portion of each curve.
For a given value of ϵ, consequently, those critical
combinations of stress for which the buckle wave length
is infinite are defined by a straight line of constant
k_y that starts at the end point of the curved portion
and extends indefinitely into the tension region of k_x.
This value of k_y is the value corresponding to Euler
strip buckling and is related to ϵ by the equation
(adapted from equation (A21) of reference 6)

$$\epsilon = \frac{-\pi \sqrt{k_y}}{\tan \dfrac{\pi \sqrt{k_y}}{2}}$$

In reference 7, the problem of buckling of finite
plates under combined longitudinal and transverse direct
stresses is investigated. The results given in refer-
ence 7 further substantiate the existence of the vertical
portions of the interaction curves, inasmuch as the
finite-plate interaction curves are seen to have portions
that approach vertical lines as the length-width ratio
of the plate increases. In figure 6 the interaction
curves for infinitely long plates with simply supported
and clamped edges are compared with the curves, based
on the results of reference 7, for similarly supported
plates with a length-width ratio of 4.

Interaction curves for restraint dependent on wave
length.- Interaction curves for a plate with edge restraint
dependent on the wave length of the buckles can be obtained
by a slight modification of the method outlined in the
preceding section. This modification consists in computing
a new value of ϵ to be used with each new assumed value
of b/λ; no other change is required. This method can
be applied only when the relationship between ϵ and b/λ
is known. For the special case of a sturdy stiffener,
the relationship of ϵ and b/λ is derived in refer-
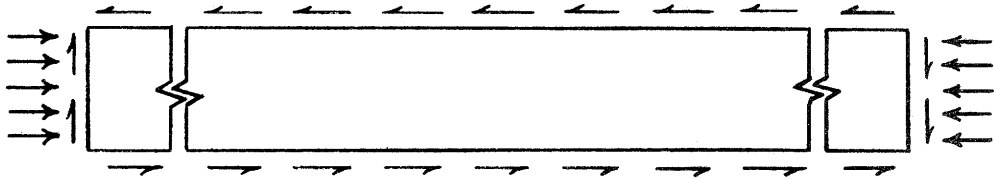ence 3.

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   Critical Stress for an Infinitely Long Flat Plate
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   Shear and Direct Stress.   NACA ARR No. 3K13, 1943.

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   NACA Rep. No. 735, 1942.

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   Hill Book Co., Inc., 1936.

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   John C.:   A Method for Determining the Column
   Curve from Tests of Columns with Equal Restraints
   against Rotation on the Ends.   NACA TN No. 903, 1943.

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   Critical Combinations of Longitudinal and Trans-
   verse Direct Stress for Flat Rectangular Plates.
   NACA ARR No. L6A05, 1946.

TABLE 1.- VALUES OF k_y WITH CORRESPONDING CRITICAL VALUES OF k_x

$\epsilon = 0$		$\epsilon = 2$		$\epsilon = 5$		$\epsilon = 10$		$\epsilon = \infty$	
k_y	k_x	k_y	k_x	k_y	k_x	k_y	k_x	k_y	k_x
1.00	0 to 2.00	1.67	0 to 2.02	2.30	0 to 2.07	2.85	0 to 2.17	4.00	0 to 2.67
.96	2.40	1.58	2.62	2.20	2.71	2.73	2.90	3.85	3.52
.89	2.67	1.24	3.31	2.00	3.16	2.48	3.42	3.50	4.23
.75	3.00	1.03	3.62	1.71	3.62	2.13	3.92	3.00	4.85
.50	3.41	.82	3.86	1.14	4.24	1.42	4.60	2.00	5.73
.25	3.73	.41	4.26	.57	4.72	.70	5.15	1.00	6.41
0	4.00	-1.64	5.66	-2.28	6.36	0	5.60	0	6.98
-1.00	4.83	-3.28	6.47	-4.56	7.33	-2.84	6.99	-4.00	8.69
-2.00	5.46	----	----	----	----	-5.68	8.06	-8.00	10.00

(a) Loading combination treated in reference 1.
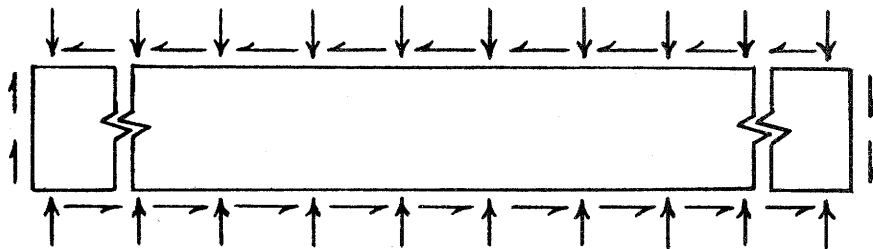
(b) Loading combination treated in reference 2.
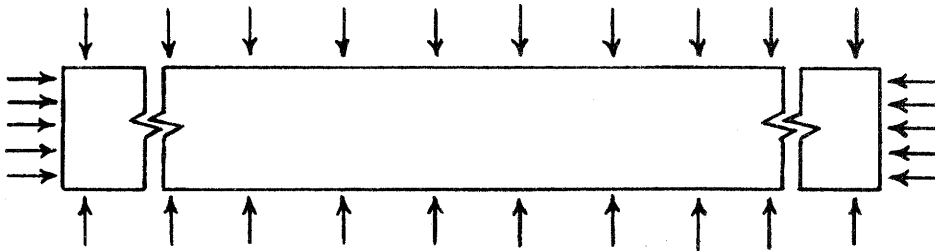
(c) Loading combination treated in present paper.

Figure 1.- Buckling of an infinitely long flat plate under
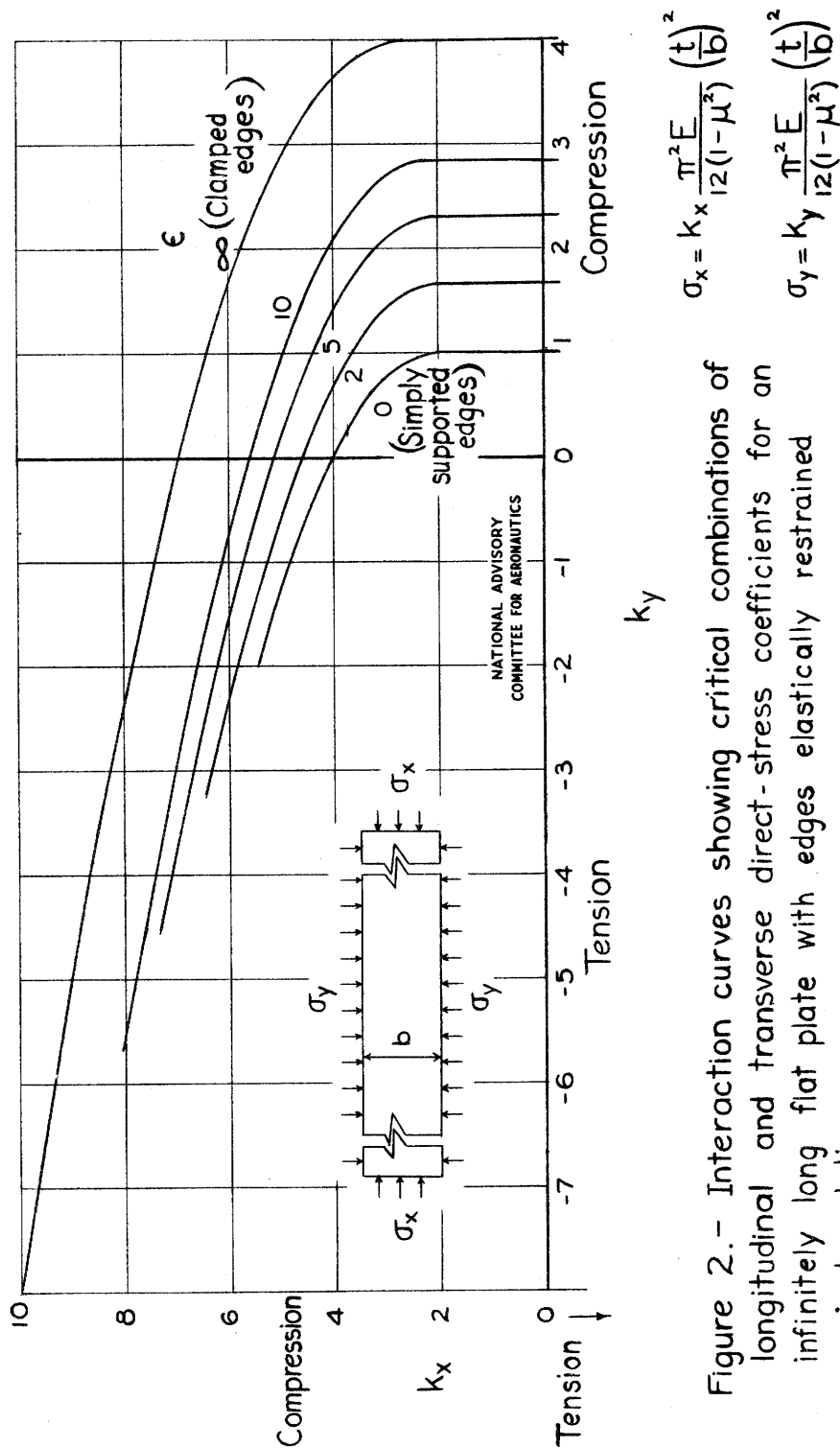combined loads.

Fig. 2　　　　　　　　　　　　　　　　　　　NACA ARR No. L6A05a

$$\sigma_x = k_x \frac{\pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

$$\sigma_y = k_y \frac{\pi^2 E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

ϵ (Clamped edges)

∞

10

5

2

0 (Simply supported edges)

Compression

Tension

k_y

Compression

Tension

k_x

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Figure 2.- Interaction curves showing critical combinations of longitudinal and transverse direct-stress coefficients for an infinitely long flat plate with edges elastically restrained against rotation.
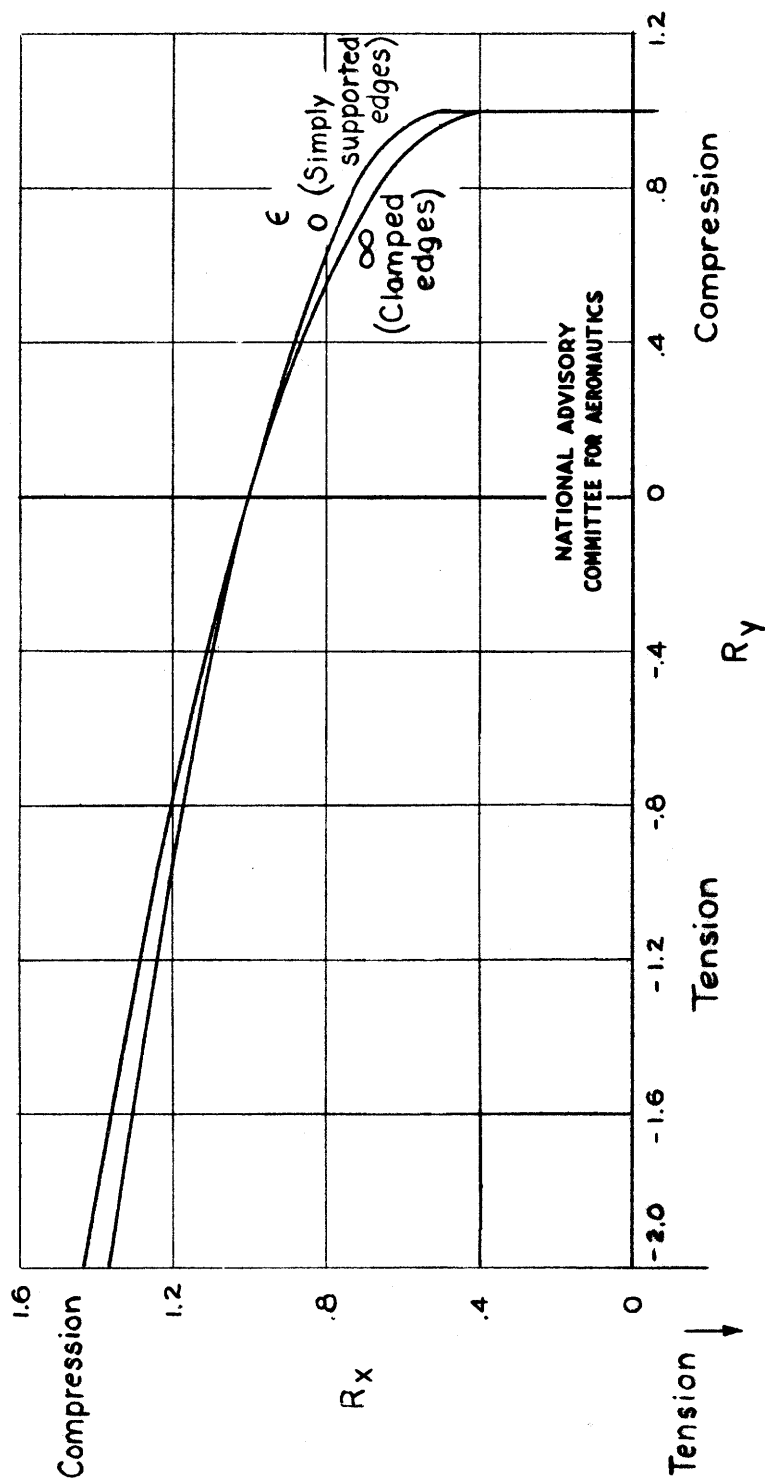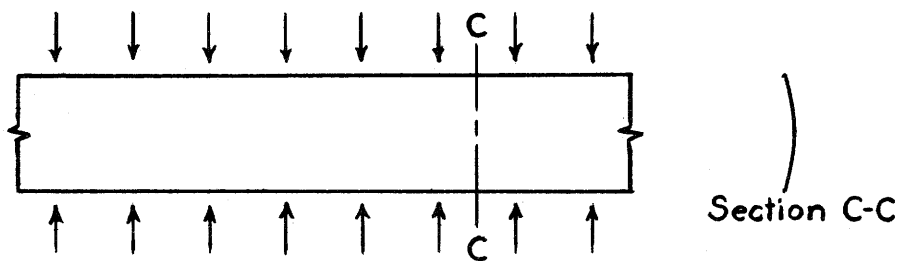
Figure 3.- Critical combinations of longitudinal direct-stress ratio R_x and transverse direct-stress ratio R_y for an infinitely long flat plate.
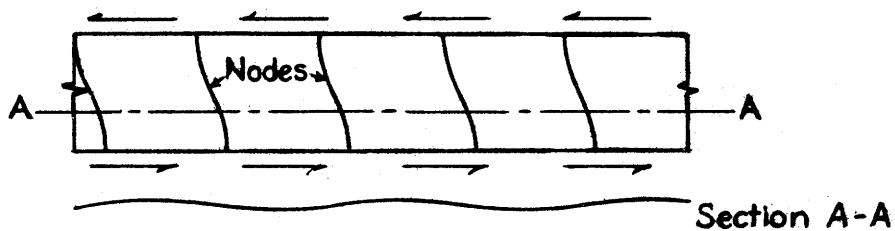
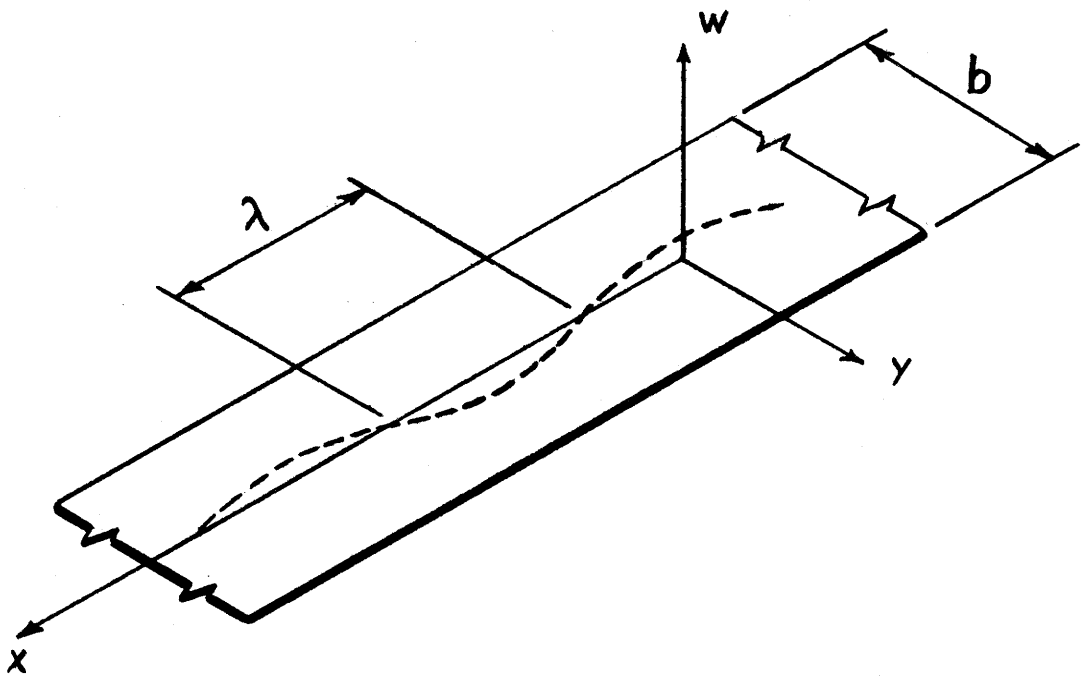(a) Buckle form for shear alone.
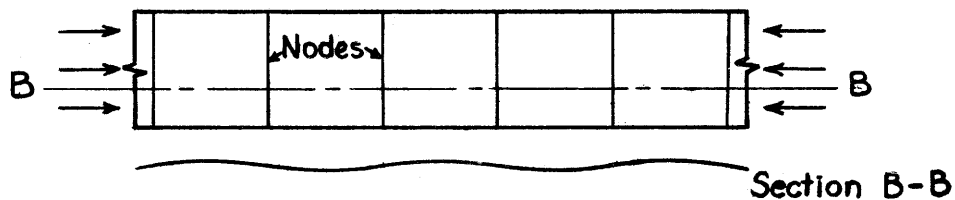
(b) Buckle form for longitudinal compression alone.

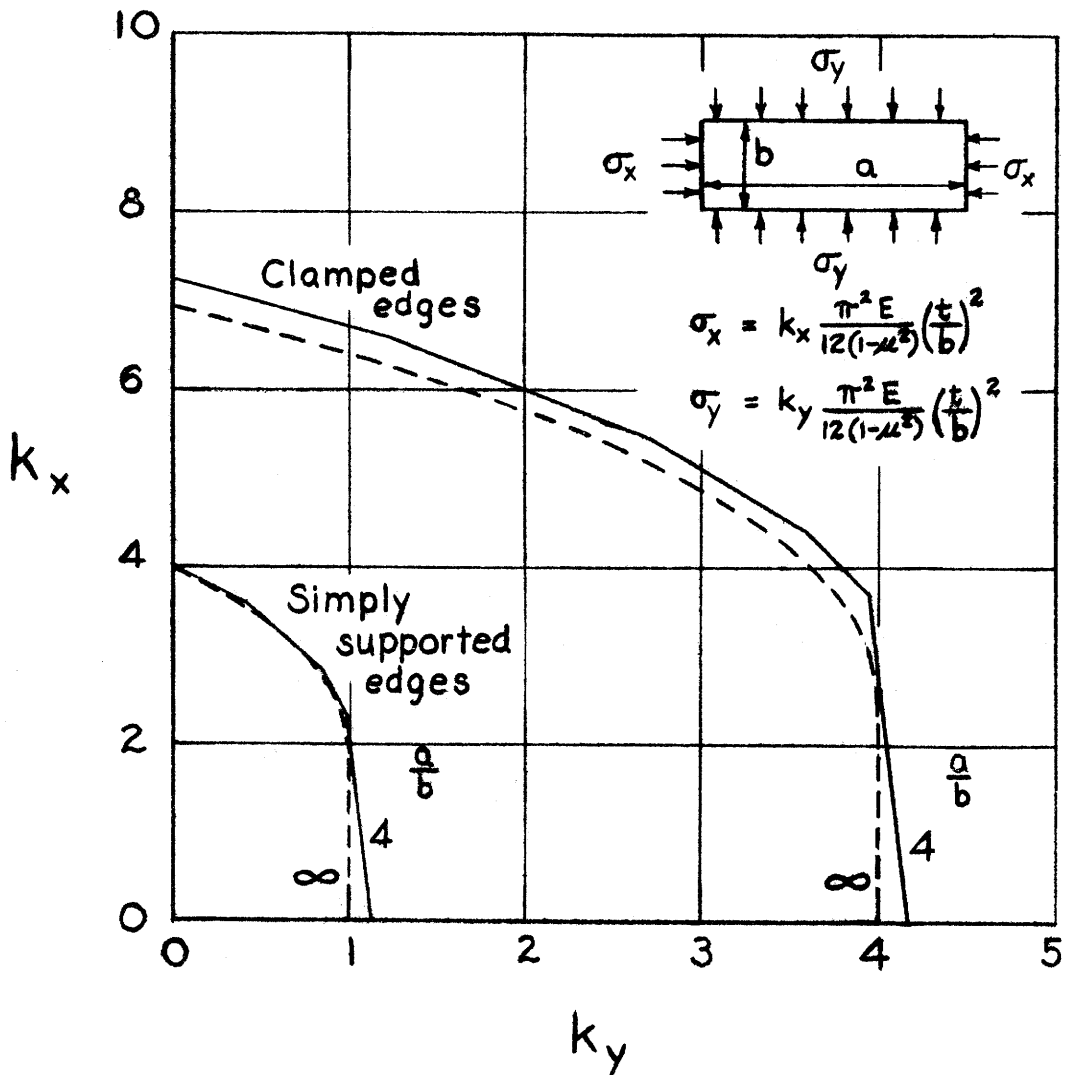(c) Buckle form for transverse compression alone.

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Figure 4.- Buckle forms for a simply supported, infinitely
long flat plate.

Figure 5.- Coordinate system used in the present paper for
an infinitely long flat plate.

Fig. 6   NACA ARR No. L6A05a

Figure 6.- Comparison of interaction curves for infinitely
long plates and interaction curves for plates having
a length-width ratio of 4.